2.004

Control Theory

$e = 2.71828$

$e^{-1} = 0.3679$

$e^{-2} = 0.1353$

$e^{-3} = 0.0498$

$e^{-4} = 0.0183$

error = Desired − actual

$AV' + BV = C(t)$

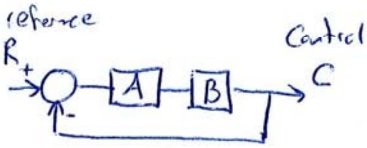
$\tau = \frac{A}{B}$

$$\frac{d^2y}{dt^2} + 2\zeta\omega_n\frac{dy}{dt} + \omega_n^2 y = 0$$

$$\mathcal{L}\{f(t)\} = \int_{0^-}^{\infty} f(t)\,e^{-st}\,dt$$

$$s = \sigma + j\omega$$

$j = \sqrt{-1}$



$\frac{C}{R} = TF = \frac{AB}{1+AB}$

$$\mathcal{L}\{f'\} = sF(s) - f(0^-)$$

$$\mathcal{L}\{f''\} = s^2F(s) - sf(0^-) - \dot f(0)$$

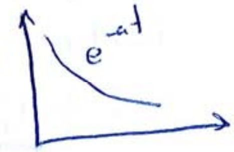
$$\mathcal{L}\left\{\frac{d^nf}{dt^n}\right\} = s^nF(s) - \sum_{K=1}^{n} s^{n-K} f^{K-1}(0)$$

$$\mathcal{L}\left\{\int_{0^-}^{t} f(\sigma)\,d\sigma\right\} = \frac{1}{s}F(s)$$

Unit Step

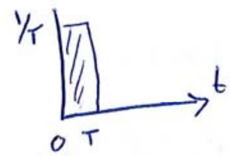
$U_s(s) = \int_{0^-}^{\infty} 1\,e^{-st} = \frac{1}{s}$

One sided exp $f(t) = e^{-at}$



$F(s) = \int_{0^-}^{\infty} e^{-at}e^{-st} = \frac{1}{s+a}$

short Impulse



$F(s) \approx 1$

State Space

$\underline{\dot X} = \underline{A}\,\underline{x} + \underline{B}\,\underline{U}$ input

$\underline{Y} = \underline{C}\,\underline{x} + \underline{D}\,\underline{U}$ output

to ↓ transform Function

Final Value Th^m

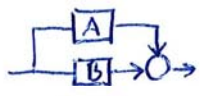
$$\lim_{t\to\infty} f(t) = \lim_{s\to 0}\{sF(s)\}$$

use to compute steady state

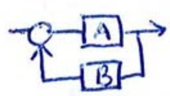
$\underline{s}\,\underline{X}(s) = \underline{A}\,\underline{X}(s) + \underline{B}\,\underline{U}(s)$

$X(s) = (sI - A)^{-1}BU(s)$

$$\boxed{= C(sI-A)^{-1}B + D}$$

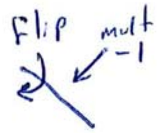


A + B

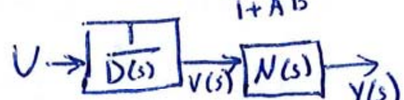


$\frac{A}{1+AB}$

Matrix Manipulation $A^{-1} = \frac{\mathrm{adj}[A]}{\det[A]}$



$$\underline{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \to \mathrm{adj}\,\underline{A} = \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$



$$V(s) = \frac{1}{a_ns^n + a_{n-1}s^{n-1} + \dots + a_1s + a_0}U(s)$$

$N(s) = b_ms^m + b_{m-1}s^{m-1} + \dots + b_0$

$\underline{Y} = \underline{C}\,\underline{X} + \underline{D}\,\underline{U}$

$$= \begin{bmatrix} b_0 & b_1 & b_2 & \dots \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ \vdots \\ X_n \end{bmatrix} + \begin{bmatrix} \ \end{bmatrix}U$$

$$\begin{bmatrix} \dot X_1 \\ \dot X_2 \\ \vdots \\ \dot X_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & \\ 0 & 0 & 0 & \dots & 0 \ \ 1 \\ -\frac{a_0}{a_n} & -\frac{a_1}{a_n} & & \dots & -\frac{a_{n-1}}{a_n} \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1/a_n \end{bmatrix}U$$

"Phase Variable" Form of State Variables.

Laplace Transforms

$\delta(t) \to 1$

$u(t) \to 1/s$

$t\,u(t) \to 1/s^2$

$t^n u(t) \to \frac{n!}{s^{n+1}}$

$e^{-at}u(t) \to \frac{1}{s+a}$

$\sin\omega t\,u(t) \to \frac{\omega}{s^2+\omega^2}$

$\cos\omega t\,u(t) \to \frac{s}{s^2+\omega^2}$

$\frac{X(s)}{U(s)} = \frac{1}{s^3 + 6s^2 + 5s + 10K}$

$\frac{Y(s)}{U(s)} = 10Ks + 10B$

$$\dot{\underline{X}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10K & -5 & -6 \end{bmatrix}\underline{X} + \begin{bmatrix} 0 \\ 0 \\ 1/1 \end{bmatrix}U(t)$$

$$Y = \begin{bmatrix} 10B & 10K & 0 \end{bmatrix}\begin{bmatrix} X \\ \dot X \\ \ddot X \end{bmatrix} + [0]\,U(t)$$

low → High phase Variable State space Representat

HADTT

Velocity	mass	Dampers	Springs	Forces
	$\dot{V}_m = \frac{1}{m} \underline{F_m}$	$V_B = \frac{1}{B} F_B$	$\dot{F}_K = K \underline{V_K}$	
	node	$F_B = B V_B$	loop add K Σ loop	

Voltage	Capacitors	Resistors	Inductors	Currents
	$\dot{V}_C = \frac{1}{C} \underline{i_C}$	$V_{RL} = R\, i_{RL}$	$\dot{i}_L = \frac{1}{L} \underline{V_L}$	
	node	$i_{Rs} = \frac{1}{R_s} V_{Rs}$	loop	

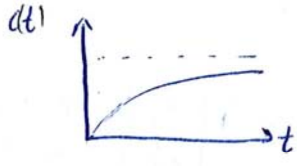
∡ Vel	Inertia	Friction	Springs	Torques
	$\dot{\Omega}_J = \frac{1}{J} \underline{T_J}$	$\Omega_B = \frac{1}{B} T_B$	$\dot{T}_K = K \underline{\Omega_K}$	
	node	$T_B = B \Omega_B$	loop	

Pressure	Inertia	Friction	Moving Mass	Flow Rate
	$\dot{P}_{Cf} = \frac{1}{C_f} \underline{Q_{Cf}}$	$P = R_f Q$	$\dot{Q} = \frac{1}{I_f} \underline{P}$	
	node	$Q = \frac{1}{R_f} P$	loop	

Overdamped responses

Eigen Value: $-\sigma_1, -\sigma_2$

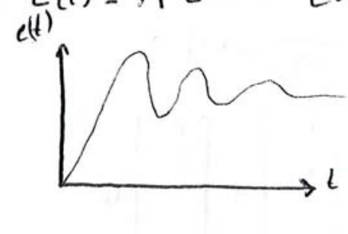
$$c(t) = K_1 e^{-\sigma_1 t} + K_2 e^{-\sigma_2 t}$$



Underdamped responses

Eigen Values: $-\sigma_d \pm j\omega_d$

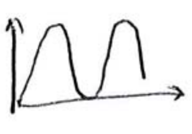
$$c(t) = A e^{-\sigma_d t} \cdot \cos(\omega_d t - \phi)$$



Undamped Responses

Eigen Value $\pm j\omega_1$

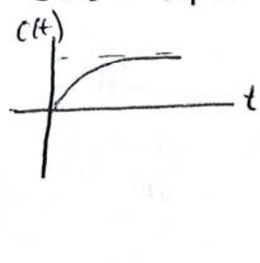
$$c(t) = A \cos(\omega_1 t - \phi)$$



Critically damped

Eigen Value $-\sigma_1, -\sigma_1$

$$c(t) = K_1 e^{-\sigma_1 t} + K_2 t e^{-\sigma_1 t}$$



$$\omega_n^2 = \frac{1}{LC}$$

$$G(s) = \frac{b}{s^2 + as + b}$$

$$\omega_n = \sqrt{b}$$

$$\zeta = \frac{a/2}{\omega_n} = \frac{a}{2\sqrt{b}}$$

$$T_p = \frac{\pi}{\omega_n (1-\zeta^2)^{1/2}}$$

time to reach 1st peak

$$\zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}}$$

$$\%OS = e^{-(\zeta\pi/\sqrt{1-\zeta^2})} \times 100$$

percent Over shoot.

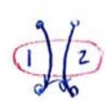
$$T_s = \frac{4}{\zeta \omega_n}$$

Settling time to reach ±2% Steady state

2.004 S2

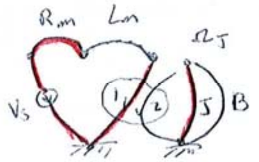
$P_1 = f_1 V_1$ $\quad P_2 = f_2 V_2$

$P_1 + P_2 = 0$



Transformer
$V_1 \propto V_2$
$f_1 \propto f_2$
one of each.

Gyrator
$V_1 \propto f_2$
$f_1 \propto V_2$
Both in or Both out



$\dot\Omega_J = \frac{1}{J} T_J$ $\quad T_J = -T_B - T_2$

$\dot i_{Lm} = \frac{1}{L} V_{Lm}$ $\quad V_{Lm} = V_s - V_{Rm} - V_1$

$V_{Rm} = R_m i_{Rm}$ $\quad i_{Rm} = i_{Lm}$

$T_B = B \Omega_B$ $\quad \Omega_B = \Omega_J$

$V_1 = \frac{1}{K_a} \Omega_2$ $\quad \Omega_2 = \Omega_J$

$T_2 = \frac{-1}{K_a} i_1$ $\quad i_1 = i_{Lm}$

$G(s) = \frac{N(s)}{D(s)}$

$D(s) \Rightarrow \det(sI - A) = 0$

Normaltree
1. Access Variable Sources
2. A type elements
3. Twoports elements
4. D type elements
5. T type elements
6. No through var sources

TF. $\quad G(s) = \dfrac{b_m s^m + \dots b_1 s + b_0}{a_n s^n + \dots + a_1 s + a_0}$

$$\rightarrow \boxed{b_m s^m + \dots b_1 s + b_0} \xrightarrow{X(s)} \boxed{\frac{1}{a_n s^n + \dots + a_1 s + a_0}} \rightarrow Y(s)$$

$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \dots + a_1 \frac{dy}{dt} + a_0 y = X(t)$ ode

solve ode

Charc. Eqn: $(a_n \lambda^n + a_{n-1} \lambda^{n-1} + \dots + a_1 \lambda + a_0) = 0$

$Y_h(t) = \sum_{k=1}^{n} C_k e^{\lambda_k t}$

Method of Undetermined Coef

$U(t)$	$Y_p(t)$
K	K
Kt^n	$K_n t^n + K_{n-1} t^{n-1} + \dots$
$Ke^{\lambda t}$	$Ke^{\lambda t}$
$Ke^{j\omega t}$	$Ke^{j\omega t}$
$K\cos(\omega t)$ $K\sin(\omega t)$	$K_1 \cos\omega t + K_1 \sin\omega t$

Euler

$\cos(\omega t) = \frac{1}{2}\left(e^{j\omega t} + e^{-j\omega t}\right)$

$\sin(\omega t) = \frac{1}{2j}\left(e^{j\omega t} - e^{-j\omega t}\right)$

$\lambda = a \pm jb$

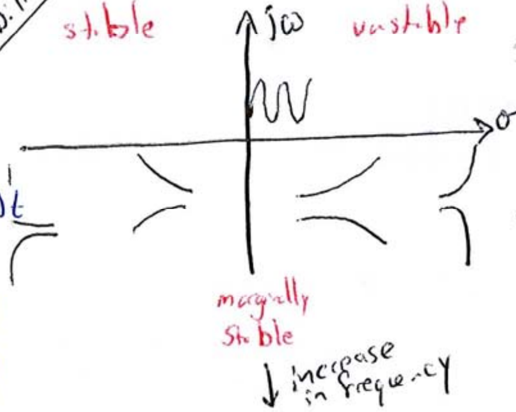
$(a + j\omega) e^{(\sigma + j\omega)t} + (a - jb) e^{(\sigma - j\omega)t}$

$2ae^{\sigma t}\cos(\omega t) + 2be^{\sigma t}\sin(\omega t)$

$Y(t) = 2\sqrt{a^2+b^2}\, e^{\sigma t} \sin(\omega t + \phi)$

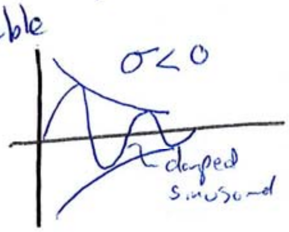
$\phi = \tan^{-1}(a/b)$

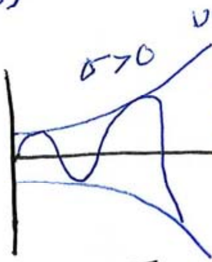
stability



Stability only affected by poles of system.

- All pole in left half plane.
- Zeros affect rise time





$\zeta = \dfrac{\text{exponential decay frequency}}{\text{Natural frequency}}$

$G(s) = \dfrac{b}{s^2 + as + b}$; $G(s) = \dfrac{b}{s^2 + b}$

$\omega_n = \sqrt{b}$

$\zeta = \dfrac{a/2}{\omega_n} = \dfrac{a}{2\sqrt{b}}$

$T_p = \dfrac{\pi}{\omega_n \sqrt{1 - \zeta^2}} = \dfrac{\pi}{\omega_d}$

$5\% \sim \frac{\sqrt{2}}{2} = \zeta$

0.707

$\%OS = e^{-(\zeta\pi / \sqrt{1-\zeta^2})} \times 100$

$\dfrac{K}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

$C_{final} = \dfrac{K}{\omega_n^2}$

$G(s) = \dfrac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

$\zeta = \dfrac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}}$

$G(s) = \dfrac{b_n s^n + \dots + b_1 s + b_0}{a_n s^n + \dots + a_1 s + a_0}$

$= K + \dfrac{N'(s)}{D(s)}$ $\quad K = \dfrac{b_n}{a_n}$

$\pm 2\%$

$T_s = \dfrac{-\ln(0.02\sqrt{1-\zeta^2})}{\zeta\omega_n}$

$G(s) = \dfrac{s+b}{s+a} = 1 - \dfrac{b-a}{s+a}$

$T_s = 4/\zeta\omega_n$

$G(s) = \dfrac{ab}{c} \dfrac{s+c}{(s+a)(s+b)}$

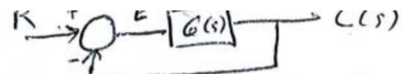
$Y_{step} = U_s(t) + \dfrac{b-a}{a}\left(1 - e^{-at}\right)$

$Y_{step} = 1 - \dfrac{b(a-c)}{c(a-b)} e^{-at} + \dfrac{a(b-c)}{c(a-b)} e^{-bt}$

$\omega_n T_r = 1.76\zeta^3 - 0.417\zeta^2 + 1.039\zeta + 1$ -

w/ Velocity feedback → ss error finite w/ prop control
0 w/ PI

with position feedback - s.s error with prop control = 0



$C(s) = G(s)E(s)$

$E(s) = R(s) - C(s)$

$\frac{E(s)}{R(s)} = \frac{1}{1+G(s)}$

$r(t) = U_s(t)\, t^{q}$

$q = 0$ step
$q = 1$ ramp
$q = 2$ parabola

Type = # poles at origin.

R

Error

	type 1	type 2	type 3
Step	C $\frac{1}{1+K_p}$	0 $K_p=\infty$	0 $K_p=\infty$
Ramp	∞ $K_v=0$	C $\frac{1}{K_v}$	$K_v=\infty$ 0
Parabola	∞ $K_a=0$	∞ $K_a=0$	C $\frac{1}{K_a}$

$\omega = 2\pi f$

Step	$1/s$	C
Ramp	$1/s^2$	t
Parabola	$1/s^3$	$\frac{1}{2}t^2$

$E(s) = \frac{R(s)}{1+G(s)}$

$e(\infty) = \lim_{s\to 0} \frac{sR(s)}{1+G(s)}$

Static error constants

position $K_p = \lim_{s\to 0} G(s)$

velocity $K_v = \lim_{s\to 0} sG(s)$

accel $K_a = \lim_{s\to 0} s^2 G(s)$

$e_{step} = \frac{1}{1+\lim_{s\to 0} G(s)}$ $\quad \frac{1}{1+K_p}$

$e_{ramp} = \frac{1}{\lim_{s\to 0} sG(s)}$ $\quad \frac{1}{K_v}$

$e_{parabola} = \frac{1}{\lim_{s\to 0} s^2 G(s)}$ $\quad \frac{1}{K_a}$

Stability

① no sign changes

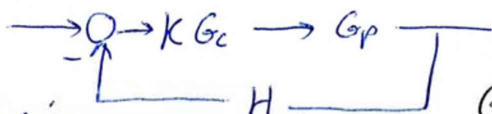
$a_3 s^3 + a_2 s^2 + a_1 s + a_0$

$a_1 a_2 > a_3 a_0$

② Routh Hurwitz

s^n	a_n	a_{n-2}	a_{n-4}	
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	...
s^{n-2}	b_1	b_2	b_3	
$\vdots$	c_1	c_2	c_3	
s^0				

Root Locus:



$G = G_c G_p H$

$G_{OL} = K G_c G_p H$

$CL = \frac{K G_c G_p}{1 + K G_c G_p H}$

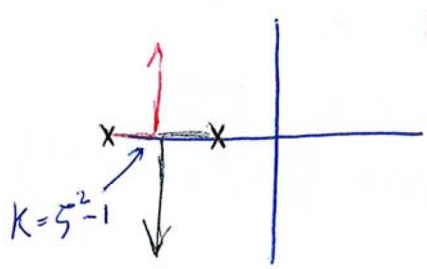
CL char equ

$1 + KG(s) = 0$

① $|KG(s)| = 1$

② $\angle G(s) = (2n+1)\pi$

$|G(s)| = 1/K$



$b_1 = \frac{-1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix}$

$b_2 = \frac{-1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix}$

$c_1 = -\frac{1}{b_1} \begin{vmatrix} a_{n-1} & a_{n-3} \\ b_1 & b_2 \end{vmatrix}$

③ Assume marginal stability

$D(s) = (s+a)(s+j\omega)(s-j\omega)$

$= s^3 + as^2 + \omega^2 s + a\omega^2$